3/3/20

Suppose $N \sim P(\lambda)$

$$p_k = \frac{e^{-\lambda} \cdot \lambda^k}{k!}$$

$$\Longrightarrow \frac{p_k}{p_{k-1}} = \frac{\cancel{e^{-\lambda}} \cdot \cancel{\lambda^{k}}^{\,\lambda}}{\cancel{k!}_{\,k}} \cdot \frac{\cancel{(k-1)!}}{\cancel{e^{-\lambda}} \cdot \cancel{\lambda^{k-1}}} = \frac{\lambda}{k}$$

Suppose $N \sim NB(r, \beta)$

$$p_k = \frac{r(r+1)\cdots(r+k-1)\cdot\beta^k}{k!\,(1+\beta)^{r+k}}$$

$$\Longrightarrow \frac{p_k}{p_{k-1}} = \frac{\cancel{r(r+1)\cdots}(r+k-1)\cdot\cancel{\beta^k}^{\,\beta}}{\cancel{k!}_{\,k}\,(1+\beta)^{\cancel{r+k}}} \cdot \frac{\cancel{(k-1)!}\,\cancel{(1+\beta)^{r+k-1}}}{\cancel{r(r+1)\cdots(r+k-2)}\cdot\cancel{\beta^{k-1}}}$$

$$= \frac{(r+k-1)\cdot\beta}{k(1+\beta)} = \frac{\beta}{1+\beta} + \frac{\left[\frac{(r-1)\cdot\beta}{1+\beta}\right]}{k}$$

Suppose $N \sim Bin(m, q)$

$$p_k = \binom{m}{k} q^k (1-q)^{m-k} = \frac{m!}{k!\,(m-k)!}\cdot q^k \cdot (1-q)^{m-k}$$

$$\Longrightarrow \frac{p_k}{p_{k-1}} = \frac{\cancel{m!}}{\cancel{k!}_{\,k}\,\cancel{(m-k)!}} \cdot \cancel{q^k}^{\,q} \cdot \cancel{(1-q)^{m-k}} \cdot \frac{\cancel{(k-1)!}\cdot\cancel{(m-k+1)!}^{\,m-k+1}}{\cancel{m!}\cdot\cancel{q^{k-1}}\,(1-q)^{\cancel{m-k+1}}}$$

$$= \frac{(m-k+1)\cdot q}{k(1-q)} = \frac{-q}{1-q} + \frac{\left[\frac{(m+1)\cdot q}{1-q}\right]}{k}$$

M255: The $\underline{(a,b,0) \text{ class}}$ of discrete distributions is defined by the property

$$\frac{p_k}{p_{k-1}} = a + \frac{b}{k} \quad (\text{start with } p_0)$$

$\therefore$ Given p_0,

$$p_1 = \left(a + \frac{b}{1}\right) \cdot p_0$$
$$p_2 = \left(a + \frac{b}{2}\right) \cdot p_1$$
$$p_3 = \left(a + \frac{b}{3}\right) \cdot p_2$$
$$\vdots$$

Remarks: (See previous page)

1) $P(\lambda)$ is $(a,b,0)$ with $a = 0$ and $b = \lambda$
2) $NB(r,\beta)$ is $(a,b,0)$ with $a = \frac{\beta}{1+\beta}$ and $b = \frac{(r-1)\cdot\beta}{1+\beta}$
3) $B(m,q)$ is $(a,b,0)$ with $a = \frac{-q}{1-q}$ and $b = \frac{(m+1)\cdot q}{1-q}$
4) See Tables for a & b values

Note: These distributions ($P(\lambda)$, $NB(r,\beta)$, and $Bin(m,q)$) make up the entire $(a,b,0)$ class of distributions

If $a = 0$, we have $P(\lambda)$

If $a < 0$, we have $Bin(m,q)$

If $a > 0$, we have $NB(r,\beta)$

Now let's modify an $(a, b, 0)$ distribution by changing the p_0 starting value.

<u>Notation</u>: $N \sim (a,b,0)$ class

$N^M \sim$ the modified distribution with probabilities

p_0^M = changed value of p_0 (given)

$\left.\begin{matrix} p_1^M \\ p_2^M \\ \vdots \end{matrix}\right\}$ The other p_k values also change, but we require the relative magnitude between the values remains unchanged

$$\Longrightarrow p_k^M = c \cdot p_k$$

<u>Q</u>: $c = ?$

$$1 = p_0^M + p_1^M + p_2^M + \cdots = p_0^M + c \cdot p_1 + c \cdot p_2 + \cdots$$

$$= p_0^M + c \cdot (\underbrace{p_1 + p_2 + \cdots}_{= 1 - p_0})$$

$$\therefore \; c = \frac{1 - p_0^M}{1 - p_0}$$

We have two distributions now; N & N^M

N	Pr
0	p_0
1	p_1
2	p_2
3	p_3
$\vdots$	$\vdots$

$\Sigma = 1$

N^M	Pr
0	p_0^M = given
1	$p_1^M = c \cdot p_1$
2	$p_2^M = c \cdot p_2$
3	$p_3^M = c \cdot p_3$
$\vdots$	$\vdots$

$\Sigma = 1$

$$c = \frac{1 - p_0^M}{1 - p_0}$$

Remarks:

1) Note $\dfrac{p_k^M}{p_{k-1}^M} = \dfrac{p_k}{p_{k-1}} = a + \dfrac{b}{k}$ starting with p_1

we N^M is called an $(a, b, 1)$ distribution

2) The zero-truncated distribution is ~~obtaining~~ obtained by setting $p_0^M = 0$

3) The zero-modified Poisson distribution is not Poisson (likewise for the other $(a,b,0)$ distributions.)

4) $E[(N^M)^k] = c \cdot E[(N)^k]$

$\therefore Var(N^M) = c \cdot E[(N)^2] - c^2 \cdot (E[N])^2$

Ends Module 2: Frequency